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Canonical Proper-Time Formulation of Relativistic Particle Dynamics. II

Tepper L. Gill,[1,2] Woodford W. Zachary,[2] and James Lindesay[3]

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Our purpose in this paper is to provide the framework for a generalization of classical mechanics and electrodynamics, including Maxwell's theory, which is simple, technically correct, and requires no additional work for the quantum case. We first show that there are two other definitions of proper-time, each having equal status with the Minkowski definition. We use the first definition, called the proper-velocity definition, to construct a transformation theory which fixes the proper-time of a given physical system for all observers. This leads to a new invariance group and a generalization of Maxwell's equations left covariant under the action of this group. The second definition, called the canonical variables definition, has the unique property that it is independent of the number of particles. This definition leads to a general theory of directly interacting relativistic particles. We obtain the Lorentz force for one particle (using its proper-time), and the Lorentz force for the total system (using the global proper-time). Use of the global proper-time to compute the force on one particle gives the Lorentz force plus a dissipative term corresponding to the reaction of this particle back on the cause of its acceleration (Newton's third law). The wave equation derived from Maxwell's equations has an additional term, first order in the proper-time. This term arises instantaneously with acceleration. This shows explicitly that the long-sought origin of radiation reaction is inertial resistance to changes in particle motion. The field equations carry intrinsic information about the velocity and acceleration of the particles in the system. It follows that our theory is not invariant under time reversal, so that the existence of radiation introduces an arrow for the (proper-time of the) system.

[1] Institute for Advanced Study, Princeton, New Jersey 08540; e-mail: gill@ias.edu. On leave from the Theoretical Physics Division, Institute for Basic Research, and the Department of Mathematics, Howard University, Washington, D.C. 20059.
[2] Department of Electrical Engineering, Howard University, Washington, D.C. 20059.
[3] Department of Physics, Howard University, Washington, D.C. 20059.

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1. INTRODUCTION

1.0. Background

A major problem in the foundations of quantum field theory has been the identification of a mathematical meaning for the Feynman time-ordered operator calculus, which was developed in the 1950s for the study of quantum electrodynamics (Feynman, 1949, 1950, 1951). The idea was to deal directly with the solutions to the equations describing the system, rather than the equations themselves. This approach requires an overall space-time point of view, which means that we view a physical event as occurring on a film in which we are more aware of the outcome as more of the film comes into view. This led Feynman to consider time histories (paths) as the primary objects of physical concern. *He noticed that the major difficulty in handling complex expressions was due to a "mathematical convention"; namely, that position on paper identifies "when" (a physical process) an operator operates.* He showed that letting time act as both a physical quantity and as an index to determine the order of operators in a product permits considerable ease in operator manipulation.

Although Dyson (1949) proved the equivalence of Feynman's formulation of quantum electrodynamics to that of Schwinger and Tomonaga, Feynman's approach was both highly physical and easy to use [see Schweber (1986) for a historical review]. Since then, this approach has provided new mathematical and physical insights, and is the method of choice in many branches of physics. This method is also popular because it has the additional advantage of providing a direct relationship with our conscious view of the world. Feynman's approach raises the question as to whether or not it is possible to construct mathematical representations of physical reality which correspond to the way the world appears to us.

It was noted by Dirac (1963) that "the picture with four-dimensional symmetry does not give us the whole situation. . . ." Quantum theory has taught us that we must take a three-dimensional section of what appears to our consciousness at one time (an observation) and relate it to another three-dimensional section at another time. In reviewing attempts to merge gravitation with quantum theory, Dirac goes on to question the fundamental nature of the four-dimensional requirement in physics and notes that in some cases physical descriptions are simplified when one departs from it.

In a series of papers (Gill, 1981, 1983; Gill and Zachary, 1987), we were able to show that a natural algebraic and analytic framework known as a *Feynman–Dyson algebra* could be constructed for the implementation of Feynman's program while retaining all of its intuitive content. The methods make it possible to show (as Feynman suggested) that the path integral is a special case of the operator calculus. We were also able to prove the Dyson conjecture (Dyson, 1952) that the renormalized perturbation expansion of quantum electrodynamics does not converge, but is asymptotic. To be precise, we showed that the perturbation expansion is asymptotic in the sense of Poincaré, as the term is used in the theory of semigroups of operators (Hille and Phillips, 1957).

Our successful construction of the Feynman time-ordered operator calculus, its intuitive physical content, and our ability to prove the Dyson conjecture convinced us that the use of time as a fourth coordinate may well be a major cause of problems in attempts to merge relativity with quantum mechanics.

With the above motivation, we decided to go back and take another look at the foundations of the theory of relativity with an eye toward understanding why the use of time as a fourth coordinate arose and the extent of its necessity. (This is, of course, independent of the fact that time is a fourth dimension.) It turns out, after close analysis, *that this is a third postulate introduced by Minkowski* (1909) *as the correct way to implement the first two postulates of Einstein* (1905):

1. The physical laws of nature and the results of all experiments are independent of the inertial frame of the observer.
2. The two-way speed of light (relative to all inertial observers) is constant.

We have made a change in the second postulate to more precisely reflect what is known about the speed of light (Selleri, 1993; Selleri and Goy, 1997).

1.1. Purpose

This paper is a sequel to Gill and Lindesay (1993) and Gill *et al* (1997) (see also Gill *et al*, 1994, 1996). Here, we construct a canonical proper-time theory for relativistic particle dynamics which satisfies the above two postulates and yet does not use time as a fourth coordinate. Our purpose is to provide the basics for a generalization of classical mechanics and electrodynamics, including Maxwell's theory, which is simple, technically correct, and requires no additional work for the quantum case.

In the remainder of this section, we introduce some new material. However, our primary focus is to provide enough background so as to make the paper self-contained. We discuss three definitions of velocity and three definitions of proper-time that arise in the special theory. We then derive a direct representation of the transformations that fix the proper-time of a given observed system for all observers. This new group is closely related to, but distinct from, the Lorentz group. We then give a heuristic derivation of the free particle theory using the third definition of proper-time, so that the reader may have a complete overview of our approach at an elementary (but nonrigorous) level.

In Section 2, we provide a brief introduction to the theory of isotopes as the creation of geometry by physical interaction. The presentation is elementary, including easy examples, so that the reader can see both the physics and the mathematics. This theory (Santilli, 1983) is based on the introduction of a change in the definition of the product at the Lie algebra level caused by physical interactions and induces a change in the geometric and the analytic structure at all levels.

In Section 3, we show that the isotope methods are natural in our case. We construct a theory of directly interacting particles without treating time as the fourth component of a four-vector. We first discuss the global world view of the many-particle system in which the observer can treat the system as one particle. We prove that our theory preserves the Poincaré–Cartan invariant and then show that the proper-time change of variables (canonical) is formed by a similarity action of the proper-time (contact) group on the Poincaré group. We derive the global position vector for the system, show that it satisfies the expected commutation relations, and derive a global Lorentz force equation with few restrictions. After showing that our theory has the cluster decomposition property, we use it to prove that the universe has a unique proper-clock and (canonical) Hamiltonian which is available to all observers (although the universe has no preferred reference frame).

In Section 4, we consider the (local) world view of one particle in the many-particle system, using its proper-clock. In this case, we get the expected Lorentz force whose form can be reduced to the one that our inertial observer would obtain using his clock and the standard Hamiltonian. The only difference is that the local metric is not flat in a neighborhood of the particle. We show that, within this framework, the total local energy is conserved. It was shown by Gill *et al* (1997) that the local fields produced by a charged particle have a dissipative term that depends instantaneously on the acceleration. We show that the corresponding potentials and fields carry intrinsic information about the particle's past motion and that the electromagnetic waves have an effective mass.

In Section 5, we consider the global world view of one particle using the global proper-clock. We see that at this level the particle energy is not conserved. We derive a generalized Lorentz force which contains the local Lorentz force plus an additional dissipative force representing the reaction of the particle back on the other particles in the system. We then construct the field equations from the global point of view and show that the whole system of particles lives in a heat bath of radiation. This radiation will distribute itself thoughout the domain of the system. At no point do we make any assumptions about the structure of the particles in the system. Thus, our theory is structure-independent. As in the Wheeler–Feynman theory, there is no need to consider the interaction of particles with themselves.

In Section 6, we show that our theory is not invariant under time reversal and thus introduces a natural arrow for the proper-time of the system. We conclude with a comparison of the Minkowski and proper-time approaches.

1.2. Velocity in Special Relativity

As noted by Lévy-Leblond (1980), velocity is one of the first examples of the process in which an empirical idea is transformed into a formal concept via mathematization. Indeed, Galileo's constructive definition of velocity was a necessary requirement for his other work. *The unique definition of velocity in Galilean relativity becomes three possibilities in special relativity*:

$$\mathbf{w}_1 = d\mathbf{x}/dt, \qquad \mathbf{u} = d\mathbf{x}/d\tau, \qquad \mathbf{w}_2 = \int_0^\tau \frac{d\mathbf{w}_1}{1 - (\mathbf{w}_1/c)^2}. \tag{1.0}$$

The velocities $\mathbf{w}_1$ and $\mathbf{u}$ are well known; $\mathbf{w}_1$ represents a definition based solely on all external measurements, and $\mathbf{u}$ represents a definition based on external measurement of distance and internal measurement of time. On the other hand, $\mathbf{w}_2$ represents a definition based solely on internal measurements. The latter may appear to lack operational meaning. However, the design of ballistic missiles with internal guidance systems operationalizes this definition using an accelerometer.

Lévy-Leblond provides a clear operational analysis of these three definitions. His work shows that each definition can be used in defining velocity and indirectly raises the physical question: *Which velocity is most useful in understanding physical systems*? By this we mean: Which of these definitions of velocity is most useful in constructing representations of physical reality that are direct, simple, and consistent with experiment?

Minkowski's introduction of time as a fourth coordinate and his discovery of proper-time has tended to link these two distinct concepts, and thus helped to cloud the issue of velocity in special relativity. There is a continuous mixture of both the first and the second definition in all aspects of the special theory and its applications.

1.3. Proper-Time

We consider three definitions for the proper-time of a physical system. *A priori*, each definition has equal status from the point of view of the special theory of relativity. To provide a framework, let us consider (for simplicity) two inertial observers X and X′ with the same orientation. We further assume that the (proper) clocks of X and X′ both begin when their origins coincide. Assuming that X′ is moving with uniform velocity **v** as seen by X, let the source of an electromagnetic field whose velocity was zero at time $t = t' = 0$ move with velocity **w** as seen by X and velocity **w**′ as seen by X′. It follows that

$$\mathbf{x}' = \mathbf{x} - \gamma(\mathbf{v})\mathbf{v}t + (\gamma(\mathbf{v}) - 1)(\mathbf{x}\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \quad t' = \gamma(\mathbf{v})(t - \mathbf{x}\cdot\mathbf{v}/c^2), \tag{1.1a}$$

$$\mathbf{x} = \mathbf{x}' + \gamma(\mathbf{v})\mathbf{v}t' + (\gamma(\mathbf{v}) - 1)(\mathbf{x}'\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \quad t = \gamma(\mathbf{v})(t' + \mathbf{x}'\cdot\mathbf{v}/c^2), \tag{1.1b}$$

with $\gamma(\mathbf{v}) = 1/[1 - (\mathbf{v}/c)^2]^{1/2}$, represent the Lorentz transformations between our two observers.

1.4. The Minkowski Definition

The first and best-known definition of proper-time is due to Minkowski (1909): $d\tau = \gamma[w(t)]\, dt$ and $d\tau = \gamma[w(t')]\, dt'$. Minkowski wrote this as

$$(d\tau)^2 = (dt)^2 - (1/c^2)(d\mathbf{x})^2 = (dt)^2[1 - (\mathbf{w}/c)^2], \qquad \mathbf{w} = d\mathbf{x}/dt; \tag{1.2a}$$

$$(d\tau)^2 = (dt')^2 - (1/c^2)(d\mathbf{x}')^2 = (dt')^2[1 - (\mathbf{w}'/c)^2], \qquad \mathbf{w}' = d\mathbf{x}'/dt'. \tag{1.2b}$$

This leads to the world-line postulate and Minkowski space. It is a natural approach to the implementation of the first two postulates using Lorentz invariance.

1.5. The Proper-Velocity Definition

A transparent way to view this case is to begin with Minkowski's definition (1.2) and rewrite it as

$$(dt)^2 = (d\tau)^2 + (1/c^2)(d\mathbf{x})^2 = (d\tau)^2[1 + (\mathbf{u}/c)^2], \qquad \mathbf{u} = d\mathbf{x}/d\tau. \tag{1.3a}$$

In this case, the observer in the **X**′ frame will have

$$(dt')^2 = (d\tau)^2 + (1/c^2)(d\mathbf{x}')^2 = (d\tau)^2[1 + (\mathbf{u}'/c)^2], \qquad \mathbf{u}' = d\mathbf{x}'/d\tau. \tag{1.3b}$$

This raises the possibility that the dynamics of special relativity can be formulated in Euclidean space provided we standardize the definition of velocity (proper-velocity of the source) for all observers. In this case, since τ is the same for all observers, there is no gain in making it a fourth coordinate. We must now, however, prove that the first two postulates are satisfied. Recently, Montanus (1997) has used (1.3a) to formulate a version of general relativity in flat space-time.

If **w** is constant, we have from (1.1) and (1.3) that $t = \delta(u)\tau$ and $t' = \delta(u')\tau$, so that

$$\mathbf{x}' = \mathbf{x} - \gamma(\mathbf{v})\mathbf{v}\delta(u)\tau + (\gamma(\mathbf{v}) - 1)(\mathbf{x}\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.4a}$$

$$\mathbf{x} = \mathbf{x}' + \gamma(\mathbf{v})\mathbf{v}\delta(u')\tau + (\gamma(\mathbf{v}) - 1)(\mathbf{x}'\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}. \tag{1.4b}$$

In the general case, **w** is not constant, so that

$$t = \int_0^\tau \delta[u(s)]\, ds, \qquad t' = \int_0^\tau \delta[u'(s)]\, ds. \tag{1.5a}$$

It follows that t (or t') is nonlocal as a function of τ in the sense that the value depends on the particular physical history (proper-time path) of the source. Setting

$$\Lambda(h) = \frac{1}{\tau}\int_0^\tau \delta[h(s)]\, ds, \tag{1.5b}$$

our transformations between observers become

$$\mathbf{x}' = \mathbf{x} - \gamma(\mathbf{v})\mathbf{v}\Lambda(u)\tau + (\gamma(\mathbf{v}) - 1)(\mathbf{x}\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6a}$$

$$\mathbf{x} = \mathbf{x}' + \gamma(\mathbf{v})\mathbf{v}\Lambda(u')\tau + (\gamma(\mathbf{v}) - 1)(\mathbf{x}'\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6b}$$

$$\mathbf{u}' = \mathbf{u} - \gamma(\mathbf{v})\mathbf{v}\delta(u) + (\gamma(\mathbf{v}) - 1)(\mathbf{u}\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6c}$$

$$\mathbf{u} = \mathbf{u}' + \gamma(\mathbf{v})\mathbf{v}\delta(u') + (\gamma(\mathbf{v}) - 1)(\mathbf{u}'\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6d}$$

$$\mathbf{a}' = \mathbf{a} - \gamma(\mathbf{v})\frac{\mathbf{v}(\mathbf{a}\cdot\mathbf{u})}{\delta(u)c^2} + (\gamma(\mathbf{v}) - 1)(\mathbf{a}\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6e}$$

$$\mathbf{a} = \mathbf{a}' + \gamma(\mathbf{v})\frac{\mathbf{v}(\mathbf{a}'\cdot\mathbf{u}')}{\delta(u')c^2} + (\gamma(\mathbf{v}) - 1)(\mathbf{a}'\cdot\mathbf{v}/\|\mathbf{v}\|^2)\mathbf{v}, \tag{1.6f}$$

where **a** (**a**$'$) is the particle proper-(three) acceleration. It should also be noted that, by the mean value property for integrals, we can find a unique $s(\tau)$ for each τ, with $0 < s(\tau) < \tau$, such that $u_\tau = u(\tau - s(\tau))$ and $\Lambda(u) = \delta(u_\tau)$. It is clear that this property is observer-independent since

$$t' = \gamma(v)(t - \mathbf{x}\cdot\mathbf{v}/c^2) \Rightarrow \Lambda(u') = \gamma(v)(\Lambda(u) - \mathbf{x}\cdot\mathbf{v}/c^2). \tag{1.6g}$$

With the above approach, we also provide the only rational solution to the problem of distant simultaneity. It is clear that all observers have the option of using their individual clocks with no hope of agreeing on the time occurrence of any event associated with the source. On the other hand, if each observer agrees to use the proper-clock of the source, we see from (1.4) that, although they will not agree on the proper-velocity of the source, they will always agree on the time occurrence of any event associated with the source.

1.6. The Canonical Variables Definition

The canonical definition is the only one of the three that is independent of the number of particles. It originated in Gill and Lindesay (1993) as an attempt to define proper-time in quantum theory. The problem is that, although $dt/d\tau$ makes sense in the classical case, it becomes an operator upon quantization. To see this, use the fact that

$$H = mc^2[1 - (\mathbf{w}/c)^2]^{-1/2} = \sqrt{c^2\mathbf{p}^2 + m^2c^4} \Rightarrow \tag{1.7}$$

$$d\tau = (mc^2/H)\, dt = mc^2(\sqrt{c^2\mathbf{p}^2 + m^2c^4})^{-1/2}\, dt. \tag{1.8}$$

It was shown that this approach can work if we treat the transformation at the classical level as a canonical change of variables. To see how this is done in the free case, let W be any classical observable so that the Poisson bracket defines Hamilton's equations in the X frame by

$$\frac{dW}{dt} = \frac{\partial H}{\partial \mathbf{p}}\frac{\partial W}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}}\frac{\partial W}{\partial \mathbf{p}} = \{H, W\}, \tag{1.9a}$$

so that:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{x}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad H = \sqrt{c^2\mathbf{p}^2 + m^2c^4}. \tag{1.9b}$$

Next, using $dt = (H/mc^2)\, d\tau$, we find the time evolution of the function W by the chain rule:

$$\frac{dW}{d\tau} = \frac{dW}{dt}\frac{dt}{d\tau} = \frac{H}{mc^2}\{H, W\}. \tag{1.10a}$$

The energy functional K conjugate to the proper time τ must therefore satisfy

$$\{K, W\} = \frac{H}{mc^2}\{H, W\}. \tag{1.10b}$$

The most general solution is

$$K = mc^2 + \int_{mc^2}^{H} (dt/d\tau)\, dH' = mc^2 + \int_{mc^2}^{H} (H'/mc^2)\, dH'. \tag{1.11}$$

If the mass m is fixed and we allow the Lorentz frame to vary, we get

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2} = \frac{\mathbf{p}^2}{2m} + mc^2. \tag{1.12}$$

This form of the Hamiltonian looks like the nonrelativistic case, but is fully relativistic and eliminates the problems associated with the troubling square root in the standard implementation. If we hold $\mathbf{P} = \mathbf{P}_0$ fixed and allow the Lorentz frame H and m to vary, we get ($H\,dH = m\,dm\,c^4$)

$$K = mc^2 = \sqrt{H^2 - c^2\mathbf{P}_0^2}. \tag{1.13}$$

This then is the appropriate Hamiltonian in the constant-momentum frame. If we fix the Lorentz frame, then H/mc^2 is constant and we get

$$K = \frac{H^2}{mc^2} = \frac{\mathbf{p}^2}{m} + mc^2. \tag{1.14}$$

The form (1.14) was used by Gill (1982) to give a particle representation for the Klein–Gordon equation with positive probability density and with proper-time as an operator. Equation (1.13) is the form used to associate proper-time with the (off-shell) mass operator in parametrized relativistic quantum theories. See Aparicio *et al* (1995a, b) for a complete discussion of this case, and Gaioli and Garcia-Alvarez (1994) for a review of the problems associated with parametrized theories. Equation (1.12) has the advantage that it produces a clear relationship with the nonrelativistic case. We can prove that both (1.12) and (1.14) have generators and hence are true canonical transformations at the classical level. We have not constructed a canonical generator for (1.13). However, see Bakamjian and Thomas (1953). In the present paper, we focus on (1.12).

2. ISOTOPES AND PHYSICAL VARIABLES

The purpose of this section is to introduce a new approach to the inclusion of geometry in physics and to use this approach to construct our theory of interacting systems. The name isotope was coined by Santilli (1978, 1983, 1993b, and references therein). Briefly, an isotope can be thought of as a new way of relating the same or different physical systems due to a change in either the internal or the external environment. We do not seek the level of generality advocated by Santilli, nor does our final product fit all of Santilli's requirements. However, in the restricted domain of our objectives, we hope to make these ideas available to a larger audience. This section should be considered introductory. We therefore strongly urge the interested reader to consult the original literature. A good starting point is the recent papers by Santilli (1996) and Kadesvili (1996). For a complete treatment, see Sourlas and Tsaras (1993).

The most efficient approach is to begin with an example which reveals all the essential mathematical issues. Let us consider the Lie algebra $so(n)$ of real $n \times n$ skew-symmetric matrices with the standard product:

$$[A, B] = AB - BA \qquad \text{for } A, B \text{ in } so(n). \tag{2.1}$$

If T is a symmetric, invertible, real $n \times n$ matrix, then we can define a new product on $so(n)$:

$$[A, B]^* = A * B - B * A = ATB - BTA. \tag{2.2}$$

Since $A^t = -A$, $B^t = -B$, and $T^t = T$, it is easy to see that $[A, B]^*$ is in $so(n)$, so that $so(n)$ is closed under the new product. It is straightforward to check that the new product satisfies the Jacobi identity so that $(so(n), +, [\cdot,\cdot]^*)$ is a Lie algebra. This is a particular case of a Lie–Santilli isotopic algebra, so-named because the early study of such algebras and the continued emphasis on their importance for mathematics and physics has been championed by R. M. Santilli for the last 20 years.

It is clear that, if we change the product at the algebraic level, this implies a corresponding change in the product at other levels. In particular, if I is the identity in the standard case, so that $I \cdot I = I$, then with the new product $*$, we must find $\hat{I}$ such that $\hat{I} * \hat{I} = \hat{I}$. This implies that $\hat{I} = T^{-1}$. Let us fix T and let $A \in so(n)$. To see what this change implies for the group, we construct the universal enveloping algebra. This allows for the generalized exponentiation necessary to identify the new group. In our case, it is easy to check that

$$\begin{aligned} g(s) &= \hat{I} + sA + \frac{1}{2!}(sA) * (sA) + \frac{1}{3!}(sA) * (sA) * (sA) + \cdots \\ &= \hat{I}(\exp\{sTA\}) = (\exp\{sAT\})\hat{I} \end{aligned} \tag{2.3}$$

satisfies

$$\frac{d}{ds} g(s)|_{s=0} = A. \tag{2.4}$$

It follows that $g(s)$ is a one-parameter curve in the group associated with the new product. Let us denote the two groups and their algebras by

$$G_1 = (SO(n), \cdot), \qquad g_1 = (so(n), +, [\cdot,\cdot]), \tag{2.5}$$

$$G_2 = (SO(n), *), \qquad g_2 = (so(n), +, [\cdot,\cdot]^*). \tag{2.6}$$

Since the properties of G_1 are well-known, let us see what is new about G_2. First we note that (2.4) shows that g_2 is the algebra for G_2 and

$$\begin{aligned} g(s)^t * g(s) &= (\hat{I}\exp\{sTA\})^t * \hat{I}(\exp\{sTA\}) = \exp\{sA^tT\}\hat{I}T\exp\{sAT\}\hat{I} \\ &= \exp\{-sAT\}\exp\{sAT\}\hat{I} = \hat{I}, \end{aligned} \tag{2.7a}$$

so that:

$$g(s)^t * g(s) = \hat{I}. \tag{2.7b}$$

This means that for X in R^n:

$$(g(s) * X)^t * (g(s) * X) = X^t * X = X^t TX. \tag{2.8}$$

We now see that the isotopic change at the algebraic level induces a corresponding change in the inner product at the vector space level.

Before going further, let us consider a (more) concrete case. Take $n = 3$ and:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \Rightarrow \tag{2.9}$$

$$X^t * X = X^t TX = \mathbf{x}_1^2 - \mathbf{x}_2^2 + \mathbf{x}_3^2. \tag{2.10}$$

This means that $G_2 = (SO(3), *)$ *is isomorphic to* $(SO(2, 1), \cdot)$. Since $(SO(3), \cdot)$ is compact while $(SO(2, 1), \cdot)$ is noncompact, this is a nontrivial result. It implies that we can study noncompact groups via their isotopic relationship to the corresponding compact groups. See Sourlas and Tsaras (1993).

In order to understand the geometric and analytic sides of this example, consider the following two Hamiltonians:

$$H_1 = \frac{1}{2m} P^t P + \frac{k}{2} X^t X = \frac{1}{2m} (\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2) + \frac{k}{2} (\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2), \tag{2.11}$$

$$H_2 = \frac{1}{2m} P^t * P + \frac{k}{2} X^t * X = \frac{1}{2m} (\mathbf{p}_1^2 - \mathbf{p}_2^2 + \mathbf{p}_3^2) + \frac{k}{2} (\mathbf{x}_1^2 - \mathbf{x}_2^2 + \mathbf{x}_3^2). \tag{2.12}$$

A simple calculation shows that both H_1 and H_2 lead to Newton's equations of motion for a (3-dimensional) harmonic oscillator, $F = -kX$. Clearly, H_1 is invariant under $SO(3)$, while H_2 is invariant under $SO(2, 1)$. It is easy to see that both H_1 and H_2 are conserved, and are in involution (i.e., their Poisson bracket is zero). *This is an example of a bi-Hamiltonian structure for the oscillator.*

The above example is not of physical interest since we cannot identify the second Hamiltonian as a sum of the (physical) kinetic and potential energies of the system. A more interesting example corresponds to the same harmonic oscillator in two different media, say when H_1 corresponds to the vacuum case and the isotope H_2 corresponds to a medium whose physical properties change as a function of spatial position, direction, and time. In this case, T will be a general matrix-valued function of X and t at each point of R^3. It is clear that, in this case, the effective physical impact is to create a change in the geometric properties of the space R^3 locally (the metric changes at each point). In our theory, T arises because of the introduction of interaction. This will be discussed further in the next section.

Let H be a given Hamiltonian with Poisson bracket $\{,\}$ and let $T = T(\{x_i\}, \{p_i\})$ be a transformation such that:

$$\{H, \cdot\} = \sum_{i=1}^{n} \frac{\partial H}{\partial \mathbf{p}_i}\frac{\partial}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i}\frac{\partial}{\partial \mathbf{p}_i} \to \{H, \cdot\}_T = \sum_{i=1}^{n} \frac{\partial H}{\partial \mathbf{p}_i} T \frac{\partial}{\partial \mathbf{x}_i} - \frac{\partial \mathbf{H}}{\partial \mathbf{x}_i} T \frac{\partial}{\partial \mathbf{p}_i}. \tag{2.13}$$

Definition 2.1. We say that T generates an *isotope of physical class I* if it is symmetric, nonsingular, and leaves the equations of motions invariant.

Definition 2.2. We say that T generates an *isotope of physical class II* if it is symmetric and nonsingular.

Definition 2.3. We say that T generates an *isotope of physical class III* if it is symmetric.

Our classification is not as general as that of Kadesvili (1996), but is sufficient for our work. The physical and mathematical foundations may be found in Santilli (1993a). See also Bogoyavlenskij (1995), who arrived at a definition closely related to Definition 2.1 while studying invariant incompatible Poisson structures for completely integrable Hamiltonian systems. The physical classes II and III represent a change in the external and/or internal physical environment, and arise because of interactions. Isotopes of physical class I arise naturally in the study of the inverse problem of the calculus of variations (Santilli 1983). *These classes led Santilli to distinguish the possibilities by noting that the coordinates used should represent the physical variables available to the experimenter in his frame of reference (the Santilli principle).*

The next section is completely devoted to isotopes of class II. Let us close this section with an instructive example of an isotope of class III.

Let $T = T(t) = (a(t)x^2 + b(t)(y^2 + z^2))^{-1/2}$, $\langle \mathbf{r}, \mathbf{r}\rangle_T = T(t)(x^2 + y^2 + z^2)$, where $a(t) = 1 + 3t$ and $b(t) = 1 - t$. If we constrain our norm to satisfy $\langle \mathbf{r}, \mathbf{r}\rangle_T = 1$, for t in [0, 1], we have at $t = 0$, $a(t)x^2 + b(t)(y^2 + z^2) = (x^2 + y^2 + z^2)^2 \Rightarrow (x^2 + y^2 + z^2)(x^2 + y^2 + z^2 - 1) = 0$. This is a unit sphere (see Fig. 1). At $t = 1$ we have $((x - 1)^2 + y^2 + z^2 - 1)((x + 1)^2 + y^2 + z^2 - 1) = 0$, which gives two unit spheres (touching). Figure 1 shows a few snapshots of the continuous change.

In the next section, we will explain how a Lorentz scalar potential energy function of the type $V(t, \mathbf{r}) = -mc[1 - \sqrt{a(t)x^2 + b(t)(y^2 + z^2)}]$ can create the above geometric effect while the effective Hamiltonian is of the harmonic oscillator type.

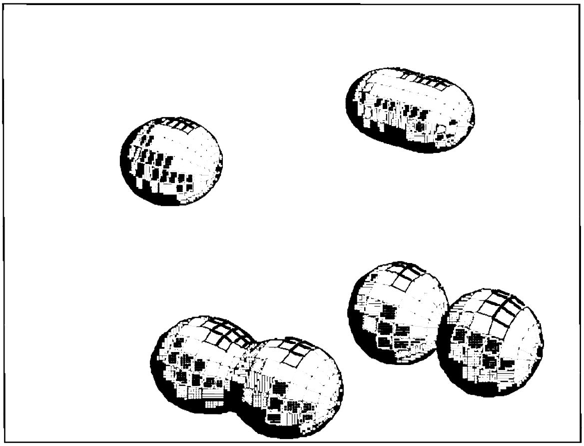

Fig. 1. Snapshots of continuous change.

3. GENERAL THEORY

3.1. Interaction

It is generally believed that the problem of interaction was resolved when the potential energy was found to fit perfectly as the scalar component of a four-vector and led to the "principle of minimal interaction." A detailed study and classification of the problems with this approach at the quantum level can be found in the book by Fushchych and Nikitin (1994). They show that only for relativistic wave equations with spin $\mathbf{s} \leq 1/2$ can one be assured that no inconsistencies occur. For $\mathbf{s} \geq 1$ we have the following types of problems (see Fushchych and Nikitin, 1994, pp. 117ff for details):

1. The equation becomes inconsistent.
2. The equation acquires redundant components making it impossible to interpret as an equation for a spin-**s** particle.
3. The equations describe faster-than-light propagation.
4. The equations become inconsistent when applied to concrete problems.

In order to introduce interaction in a consistent manner at all levels, we introduce the following:

Postulate 3.1. The proper-time Hamiltonian becomes isotopically related to the free case when interaction is turned on.

This postulate allows us to define proper-time *uniquely* at all levels by treating interaction as an isotopic deviation from the free case which causes a distortion of the local geometry of the system. We thus introduce geometry as a creation of (or by-product of) physical interactions. The important question is whether this approach reproduces what we know and provides additional insights that were not available before.

Examples. As examples, we consider the following two Hamiltonians:

$$H_1 = \sqrt{c^2\mathbf{p}^2 + (mc^2 + V)^2}, \tag{3.1a}$$

$$H_2 = H_{02} + V = \sqrt{c^2\mathbf{p}^2 + m^2 c^4} + V, \tag{3.2a}$$

corresponding to two different ways of describing particle interactions. Starting as in Gill and Lindesay (1993), we have

$$1. \qquad d\tau = \left\{\frac{mc^2 + V}{H_1}\right\} dt, \qquad \frac{dW}{d\tau} = \frac{H_1}{mc^2 + V}\{H_1, W\}; \tag{3.1b}$$

$$2. \qquad d\tau = \left\{\frac{mc^2}{H_{02}}\right\} dt, \qquad \frac{dW}{d\tau} = \frac{H_{02}}{mc^2}\{H_2, W\}. \tag{3.2b}$$

Note that in both cases $d\tau$ is derived from $d\tau = \sqrt{1 - \mathbf{w}^2/c^2}\, dt$ and hence corresponds to the standard (unique) definition of proper-time. Set

$$T_1 = \frac{mc^2}{mc^2 + V} = \frac{1}{1 + V/mc^2}, \tag{3.1c}$$

$$T_2 = \frac{H_{02}}{H_2} = I - \frac{V}{H_2}. \tag{3.2c}$$

We then have:

$$K_1 = \frac{\mathbf{p}^2}{2m} + mc^2 + V + \frac{V^2}{2mc^2}, \tag{3.1d}$$

$$\frac{dW}{d\tau} = \{K_1, W\}_{T_1} = \frac{\partial K_1}{\partial \mathbf{p}} T_1 \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial K_1}{\partial \mathbf{x}} T_1 \frac{\partial W}{\partial \mathbf{p}}, \tag{3.1e}$$

$$\{x_i, x_j\}_{T_1} = 0, \qquad \{p_i, p_j\}_{T_1} = 0, \qquad \{x_i, p_j\}_{T_1} = -\delta_{ij}T_1, \tag{3.1f}$$

$$K_2 = \frac{\mathbf{p}^2}{2m} + mc^2 + V\frac{H_{02}}{mc^2} + \frac{V^2}{2mc^2}, \tag{3.2d}$$

$$\frac{dW}{d\tau} = \{K_2, W\}_{T_2} = \frac{\partial K_2}{\partial \mathbf{p}} T_2 \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial K_2}{\partial \mathbf{x}} T_2 \frac{\partial W}{\partial \mathbf{p}}, \tag{3.2e}$$

$$\{x_i, x_j\}_{T_2} = 0, \qquad \{p_i, p_j\}_{T_2} = 0, \qquad \{x_i, p_j\}_{T_2} = -\delta_{ij}T_2. \tag{3.2f}$$

In the above two cases, the equations of motion become:

$$1. \quad \frac{d\mathbf{x}}{d\tau} = \mathbf{u} = \frac{c^2\mathbf{p}}{mc^2 + V}, \qquad \frac{d\mathbf{p}}{d\tau} = -\nabla V, \tag{3.1g}$$

$$2. \quad \frac{d\mathbf{x}}{d\tau} = \mathbf{u} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{d\tau} = -\nabla V \frac{H_{02}}{mc^2} = -\nabla V\sqrt{1 + \mathbf{u}^2/c^2}. \tag{3.2g}$$

If $mc^2 \gg V$, then $T_1 \approx I$, and if $H_2 \gg V$, $T_2 \approx$ I. As $T_1 = (1 + V/mc^2)^{-1}$ and $T_2 = (1 - V/H_2)$, if V represents an attractive potential, T_1 is nonsingular ($\neq 1/0$) as long as $V \neq mc^2$, while T_2 is nonsingular ($\neq 1/0$) as long as $V \neq H_{02}$. Note that if either becomes singular, the (induced) metric becomes meaningless. In the repulsive case, both T_1 and T_2 are always nonsingular.

The isotope T_2 arises from our implementation of the principle of minimal interaction, so the above result tells us that the curvature near particles becomes extremely distorted as they get closer and closer together. It follows that, in such cases, the physical properties in these regions will be very different from what we find when particles are far apart. At the classical level, this approach implements the principle of impenetrability; namely, that no two particles can occupy the same space at the same time. As is well-known, at the quantum level this principle is violated and results in pair production and, in our case, this corresponds to an increase in the dimension of the phase space along with other interesting physical and mathematical phenomena that will be taken up at a later time.

There is an important difference which arises here that is distinct from the methods of (differential) geometry. In the theory of symplectic manifolds, given a Poisson structure defined on the smooth functions over a phase space, $\{f, g\} = A_{ij}f^i g^j$, $f^i = \partial f/\partial y_i$, $g^j = \partial g/\partial z_j$ (summation convention), with A_{ij} skew symmetric and nonsingular, we can always find a canonical change of variables, $\mathbf{y} = \mathbf{y}(\mathbf{x}, \mathbf{p})$, $\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{p})$ (locally at each point) such that the transformed bracket represents the same geometry; and in the new variables, $\{f, g\} = J_{ij}f^i g^j$, J is the row matrix $(0, -I; I, 0)$, $f^i = \partial f/\partial p_i$, $g^j = \partial g/\partial x_j$. This is the content of Darboux's theorem (see Perelomov, 1990). In our case this is not possible since T arises on physical grounds and the variables in our theory must always be physical. To change them implies a change in the underlying physics which requires (*physical*) *justification.*

Applying our results for $V(t, \mathbf{r}) = -mc^2[1 - \sqrt{a(t)x^2 + b(t)(y^2 + z^2)}]$ $(= -mc^2[1 - r(t)])$ in equations (3.1), we can now complete the last example of Section 2. Note that K_1 becomes:

$$K_1 = \frac{\mathbf{p}^2}{2m} + \frac{mc^2}{2}(1 + r(t)^2). \tag{3.1h}$$

It is quite interesting that this is the Hamiltonian for a harmonic oscillator with a time-dependent anisotopic spring constant. The interesting physical questions raised by this example will be taken up when we extend our results to the quantum case.

3.2. Canonical Proper-Time Variables

As is well-known, attempts to construct a relativistic many-particle theory led to the no-interaction theorem. Many believed the problem was caused by the difficulty in understanding all the representations of the Poincaré group. It was pointed out by Horwitz and Piron (1973) that the same difficulty occurs in the Galilean case, where the problem is clear. The Galilean group was interpreted as the group of motion from the active point of view. This led to a representation for a free particle in the Heisenberg picture which depends on the dynamics. Piron (1972) found that the best approach in this case was to first choose the observables of the system and then construct the dynamics. He showed that this approach naturally leads to the Schrödinger picture, which is independent of the dynamics.

Our approach to the choice of variables is based on two conditions:

1. The variables must be canonical.
2. The variables must be physical.

The first condition is required (in addition to algebraic, analytic, and geometric conditions) because the origin of quantum mechanics is still mysterious and we can only be on sure footing when we quantize canonical variables. The second condition means that the variables must be operationally capable of direct measurement in experiment. This condition is clear in light of our discussion of isotopes in the last section.

The requirement of canonical variables is not as simple as it might first appear. In his work on the foundations of mechanics, Santilli (1978) identified eight different (nonequivalent) definitions of a canonical transformation, and he noted that there are many other definitions in the literature. Thus, in order to have a rational foundation for physics, we must provide a physically useful and mathematically consistent definition of a canonical transformation of variables. This issue acquires additional importance in our case since we want to transform the time variable.

Definition 3.1. A C^2 map of the variables $(\{\mathbf{x}_i\}\{\mathbf{p}_i\}, t, H) \to (\{\mathbf{X}_i\}\{\mathbf{P}_i\}, \tau, K)$ is an isocanonical contact mapping provided there are functions S, T, with $K{\bullet}d\tau = KTd\tau$ and

$$\sum_{i=1}^{n} \mathbf{p}_i \cdot \mathbf{dx}_i - H\,dt = \sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{dX}_i - K{\bullet}d\tau + dS. \tag{3.3a}$$

The following result is proven in Arnol'd (1978, p. 241).

Theorem 3.2. Let $(\{\mathbf{X}_i\}\{\mathbf{P}_i\}, \tau)$ be a coordinate system on the extended phase space $(\{\mathbf{x}_i\}\{\mathbf{p}_i\}, t)$. If functions $K(\mathbf{X}_i, \mathbf{P}_i, \tau)$, $S(\mathbf{X}_i, \mathbf{P}_i, \tau)$, and $T(\mathbf{X}_i, \mathbf{P}_i, \tau)$ exist such that

$$\sum_{i=1}^{n} \mathbf{p}_i \cdot \mathbf{dx}_i - Hdt = \sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{dX}_i - K \bullet d\tau + dS. \tag{3.3b}$$

Then the trajectories of the phase flow represented by the right-hand side are represented in the $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$ chart by the integral curves of the canonical equations:

$$\frac{d\mathbf{X}_i}{d\tau} = T\frac{\partial \mathbf{K}}{\partial \mathbf{P}_i}, \qquad \frac{d\mathbf{P}_i}{d\tau} = -T\frac{\partial \mathbf{K}}{\partial \mathbf{X}_i}. \tag{3.3c}$$

3.3. Global System Representation

We assume that there are n interacting particles which can be represented via:

$$H = \sum_{i=1}^{n} H_i = H_0 + V, \qquad H_i = H_{0i} + V_i, \tag{3.4a}$$

$$H_{0i} = \sqrt{c^2\,\boldsymbol{\pi}_i^2 + m_i^2\,c^4}, \qquad \boldsymbol{\pi}_i = \mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i, \tag{3.4b}$$

$$H_0 = \sum_{i=1}^{n} H_{0i}, \quad V = \sum_{i=1}^{n} V_i, \quad \mathbf{A} = \sum_{i=1}^{n} \mathbf{A}_i, \tag{3.4c}$$

where for our purposes, the $e_i/c\,\mathbf{A}_i$ terms may be viewed as general vector potentials to be defined in particular cases. However, we assume only direct interactions without self-interactions, so that they have the form:

$$\mathbf{A}_i = \sum_{i\neq j} \mathbf{A}_{ij}(|\mathbf{x}_i - \mathbf{x}_j|, \tau), \qquad V_i = \sum_{i\neq j} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|, \tau). \tag{3.4d}$$

Choose q (assumed $\neq 0$) so that

$$(q/c)\mathbf{A} = \sum_{i=1}^{n} (e_i/c)\mathbf{A}_i. \tag{3.4e}$$

Set $\Phi_i \equiv (1/e_i)V_i$, $\Phi \equiv (1/q)V$, $M_0 = \Sigma\, m_i$, and define M and W by

$$M_0c^2 + W = Mc^2 = \sqrt{H_0^2 - c^2\Pi^2}, \tag{3.5a}$$

where $\Pi = \mathbf{P} - (q/c)\mathbf{A}$, and $\mathbf{P}$ is the momentum of the system ($\mathbf{P} = \Sigma\,\mathbf{p}_i$). It follows that

$$W = M_0c^2 \sum_{i,j;i\neq j}^{n} \sqrt{1 + (h_{ij} - \mu_{ij})}, \tag{3.5b}$$

$$h_{ij} = \frac{1}{M_0^2c^4}(H_{0i}H_{0j} - c^2\pi_i\cdot\pi_j), \qquad \mu_{ij} = \frac{1}{M_0^2}(m_i m_j), \tag{3.5c}$$

$$H = \sqrt{c^2\Pi^2 + (M_0c^2 + W)^2} + V. \tag{3.5d}$$

Our definition of H closely corresponds to the one-particle case with minimal interaction. This differs only slightly from the Bakamjian–Thomas (1953) approach and reduces to it in the free case. We have treated the off-mass-shell part of M as a Lorentz scalar potential, and have allowed interaction to be introduced in the normal way (which they were forced to abandon). Our theory has the additional advantage that it is not constrained by the world-line postulate (which they considered unnecessary).

In this case, we take

$$d\tau = \frac{Mc^2}{H_0}dt, \qquad T = \frac{H_0}{H}\frac{M_0}{M}, \tag{3.6a}$$

$$K = \frac{H^2}{2M_0c^2} + \frac{M_0c^2}{2} = \frac{\Pi^2}{2M_0} + M_0c^2 + W + \frac{W^2}{2M_0c^2} + V\frac{H_0}{M_0c^2} + \frac{V^2}{2M_0c^2}. \tag{3.6b}$$

Assuming that the system is closed so that $dH/dt = 0$, it follows that the same is true when t is replaced by τ and H by $K(dK/d\tau = dK/dt\cdot dt/d\tau)$. We also have

$$d\tau_i = \frac{m_ic^2}{H_{0i}}dt \Rightarrow \frac{d\tau_i}{d\tau} = \frac{H_0m_i}{MH_{0i}} \tag{3.6c}$$

and, as in (3.1),

$$\frac{dW}{d\tau} = \sum_{i=1}^{n} \frac{\partial K}{\partial \mathbf{p}_i}T\frac{\partial W}{\partial \mathbf{x}_i} - \frac{\partial K}{\partial \mathbf{x}_i}T\frac{\partial W}{\partial \mathbf{p}_i} = \{K, W\}_T. \tag{3.6d}$$

Thus the total (conserved) system creates a natural geometric environment. The following result relates the phase flows for the $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, t)$ and $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$ variables.

Theorem 3.3. There exist functions $S(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$ and $T(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$ such that

$$\sum_{i=1}^{n} \mathbf{p}_i\cdot d\mathbf{x}_i - Hdt = \sum_{i=1}^{n} \mathbf{p}_i\cdot d\mathbf{x}_i - K\bullet d\tau + dS. \tag{3.7}$$

Proof. Setting $dS = (mc^2 - K)\bullet d\tau$, we have the identity $Hdt \equiv K\cdot d\tau$

$-\,dS$. It is easy to show that dS is an exact 1-form (assuming $dH/dt = 0$). Since there is no change in the $\{\mathbf{x}_i\}$, $\{\mathbf{p}_i\}$ variables, it follows from Theorem 3.2 that our transformation is (iso) canonical.

In a manner similar to that of Horwitz and Piron (1973), we can formulate a dynamical principle which generalizes Hamilton's principle using our version of the integral invariant of Poincaré–Cartan (3.7) (see Arnol'd 1978):

$$I = \oint_{\mathbf{C}} \sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - K \bullet d\tau, \tag{3.8a}$$

where $\mathbf{C}$ is a closed curve on extended (proper) phase space $\Gamma = (\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$, and the above integral is invariant for arbitrary deformations of $\mathbf{C}$ along trajectories corresponding to solutions of the equations of motion. From (3.7), we see that our approach also leads to a Lagrangian formulation. It can be written as

$$\mathbf{L}\, d\tau = \sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - K \bullet d\tau, \tag{3.8b}$$

$$\frac{d}{d\tau}\frac{\partial \mathbf{L}}{\partial \mathbf{v}_i} - T\frac{\partial \mathbf{L}}{\partial \mathbf{x}_i} = \mathbf{0}, \qquad \mathbf{v}_i = \frac{d\mathbf{x}_i}{d\tau}. \tag{3.8c}$$

Let our two inertial observers in frames X and X′ have (extended) phase space coordinates $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, t)$ and $(\{\mathbf{x}'_i\}, \{\mathbf{p}'_i\}, t')$, respectively (for the dynamics of the system of particles), and let $\mathbf{P}$ be the set of Poincaré transformations on space-time reference frames. We let $\mathbb{C}_\tau$ denote the set of canonical proper-time transformations defined on extended phase space. [Theorem 3.3 proves that $\mathbb{C}_\tau$ is a group (canonical proper-time group).] Let the map from $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, t)$ to $(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau)$ be denoted by $C[\{\mathbf{x}_i\}, t, \tau]$.

Theorem 3.4. The proper-time coordinates of the system as seen by an observer at X are related to those of an observer at X′ by the transformation

$$\mathbf{R}_{M0}[\{\mathbf{x}_i\}, \{\mathbf{x}'_i\}, \tau] = C[\{\mathbf{x}'_i\}, t', \tau]\mathbf{P}(\mathrm{X}, \mathrm{X}')C^{-1}[\{\mathbf{x}_i\}, t, \tau]. \tag{3.9}$$

Proof. The proof follows since the diagram below is commutative:

$$\begin{array}{ccc} \mathrm{X}(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, t) & \xrightarrow[\mathbf{P}]{} & \mathrm{X}'(\{\mathbf{x}'_i\}, \{\mathbf{p}'_i\}, t') \\ {\scriptstyle C[t,\,\tau]}\big\downarrow & & \big\downarrow{\scriptstyle C[t',\,\tau]} \\ \mathrm{X}(\{\mathbf{x}_i\}, \{\mathbf{p}_i\}, \tau) & \xrightarrow{\mathbf{R}} & \mathrm{X}'(\{\mathbf{x}'_i\}, \{\mathbf{p}'_i\}, \tau). \end{array}$$

The top diagram is the Poincaré map $\mathbf{P}$ from X to X′. *It is important to note that this map is between coordinates of observers.* In this sense, our approach may be viewed as a direct generalization of the standard implementation.

This means that we keep the invariance group of the underlying geometrical manifold (Poincaré), while changing the invariance group for the dynamical law to the proper-time group.

Theorem 3.4 proves that $\mathbf{R}_{M_0}$ is in the proper-time group $\mathbf{P}_\tau$, which is formed by a similarity action on the Poincaré group by $\mathbb{C}_\tau$. Since t is related to τ via a nonlocal (nonlinear) transformation, it follows that this group is not linear, and thus not covered in the Cartan classification. Fushchych and Nikitin (1987) have shown that there exist Poincaré transformations that fix the time for the free Maxwell equations (see also Fushchych and Shtelen, 1991). This result was shown to be a special case of our theory in Gill *et al* (1996).

3.6. Global Systems Dynamics

The world view of the system as a whole is obtained by using the global Hamiltonian (3.6) along with the global proper-time for the system and the global canonical variables **P** and **X** (see below). From $\mathbf{p}_i = m_i\mathbf{u}_i + (e_i/c)\mathbf{A}_i$ ($\mathbf{u}_i = d\mathbf{x}_i/d\tau_i$), we get that $\Pi = \sum m_i\mathbf{u}_i$. Using (3.6), we have

$$\mathbf{U} = \frac{d\mathbf{X}}{d\tau} = T\frac{\partial K}{\partial \mathbf{P}} = \frac{\Pi}{M}, \tag{3.10a}$$

$$\mathbf{U} = 1/M\sum_{i=1}^{n} m_i\mathbf{u}_i, \tag{3.10b}$$

$$H_0 = Mc\sqrt{\mathbf{U}^2 + c^2} = Mcb. \tag{3.10c}$$

The position for the system as a whole is implicitly defined by:

$$\mathbf{X} = \int_0^\tau \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{u}_i(\lambda)\, d\lambda + \mathbf{Y}, \tag{3.10d}$$

where **Y** is a vector with $d\mathbf{Y}/d\tau = \mathbf{0}$. Using $d\tau = Mc^2/H_0\, dt$ and $d\tau_i = m_ic^2/H_{0i}\, dt$, we see that (3.10d) reduces to ($\mathbf{v}_i = d\mathbf{x}_i/d\tau$):

$$\mathbf{X} = \int_0^\tau \sum_{i=1}^{n} \frac{H_{0i}}{H_0}\mathbf{v}_i(s)\, ds + \mathbf{Y}. \tag{3.10e}$$

Since our canonical transformation only changed t and H in the noninteracting case ($\mathbf{A}$, $V = 0$) we expect (Pauri and Prosperi, 1975):

$$\mathbf{X} = \frac{1}{\mathbf{H}_0}\sum_{i=1}^{n} H_{0i}\mathbf{x}_i(\tau) + \frac{c^2(\mathbf{S}\times\mathbf{P})}{H(Mc^2 + H)}, \tag{3.10f}$$

where **S** can be viewed as the internal spin of the system of particles and **P** is the total momentum. We thus get that (3.1f):

$$\{X_i, P_j\} = -\delta_{ij}T, \tag{3.10g}$$

as expected. Note that in this case $T = M_0/M$ [see equation (3.6a)]. It follows that the global system induces geometric effects even when the particles are not interacting. This is caused by our approach, which allows us to stay on-mass-shell during interaction. Recall that the natural canonical position operator in the many-particle case is not the vector part of a (Minkowski) four-vector. This has been a major problem for any attempt to construct a many-body relativistic theory. Since our theory does not require four-vectors for its implementation, the problem disappears. An additional advantage is that we are able to extend the principle of minimal interaction to the many-body relativistic case.

In general, each H_{0i} need not be constant, so we cannot integrate equation (3.10e). Thus, **X** is composed of the standard weighted energy term plus a term commuting with K. Assuming conservation of total energy, momentum, and angular momentum, we choose **Y** such that **X** is the *canonical center-of-mass* extended to include interactions [in the terminology of Pauri and Prosperi (1975)].

Returning to equations (3.6b) and (3.6d), it is easy to show that:

$$\frac{d\mathbf{P}}{d\tau} = -\sum_{i=1}^{n} \frac{\partial K}{\partial \mathbf{x}_i} T = \frac{b}{c} \left\{ \frac{q}{b} \left(\frac{d\mathbf{A}}{d\tau} - \frac{\partial \mathbf{A}}{\partial \tau} \right) + \frac{q}{b} \mathbf{U} \times \mathbf{B} - \nabla V \right\}, \tag{3.11a}$$

so that:

$$\frac{c}{b} \frac{d\Pi}{d\tau} = \left\{ q\mathbf{E} + \frac{q}{b} \mathbf{U} \times \mathbf{B} \right\}, \qquad \mathbf{E} = -\frac{q}{b} \frac{\partial \mathbf{A}}{\partial \tau} - \nabla \Phi. \tag{3.11b}$$

Note that we never use the global position vector **X** explicitly in computing $d\mathbf{P}/d\tau$, but equation (3.11a) implies that

$$\frac{d\mathbf{P}}{d\tau} = -\sum_{i=1}^{n} \frac{\partial K}{\partial \mathbf{x}_i} T = -T \frac{\partial K}{\partial \mathbf{X}}. \tag{3.11c}$$

If M is conserved, we get that:

$$\frac{Mc}{b} \frac{d\mathbf{U}}{d\tau} = \left\{ q\mathbf{E} + \frac{q}{b} \mathbf{U} \times \mathbf{B} \right\}. \tag{3.11d}$$

This equation expresses the global system evolution, and is a nonlinear Lorentz force (in **U**). Note that even if the global momentum is conserved ($d\mathbf{P}/d\tau = 0$), $d\mathbf{U}/d\tau$ need not be zero.

Let us return to (3.6b) and note that the observable W has another representation that relates the global system to the individual particle systems. Using $H = \sum H_j$,

$$\frac{\partial H}{\partial \mathbf{p}_i} = \sum_{j=1}^{n} \frac{\partial H_j}{\partial \mathbf{p}_i} = \sum_{j=1}^{n} \frac{m_j c^2}{H_{0j}} \frac{H_{0j}}{H_j} \frac{H_j}{m_j c^2} \frac{\partial H_j}{\partial \mathbf{p}_i} = \sum_{j=1}^{n} \frac{m_j c^2}{H_{0j}} \frac{\partial K_j}{\partial \mathbf{p}_i} T_j, \tag{3.12a}$$

$$\frac{\partial H}{\partial \mathbf{x}_i} = \sum_{j=1}^{n} \frac{\partial H_j}{\partial \mathbf{x}_i} = \sum_{j=1}^{n} \frac{m_j c^2}{H_{0j}} \frac{H_{0j}}{H_j} \frac{H_j}{m_j c^2} \frac{\partial H_j}{\partial \mathbf{x}_i} = \sum_{j=1}^{n} \frac{m_j c^2}{H_{0j}} \frac{\partial K_j}{\partial \mathbf{x}_i} T_j, \tag{3.12b}$$

where $T_j = H_{0j}/H_j$. Using (3.6c) for $d\tau_i/d\tau$, our equations of motion become

$$\frac{dW}{d\tau} = \{K, W\}_T = \sum_{j=1}^{n} \frac{d\tau_j}{d\tau} \{K_j, W\}_{T_j}, \tag{3.13a}$$

$$K_j = \frac{H_j^2}{2m_j c^2} + \frac{m_j c^2}{2}, \tag{3.13b}$$

$$\{K_j, W\}_{T_j} = \frac{dW}{d\tau_j} = \sum_{i=1}^{n} \frac{\partial K_j}{\partial \mathbf{p}_i} T_j \frac{\partial W}{\partial \mathbf{x}_i} - \frac{\partial K_j}{\partial \mathbf{x}_i} T_j \frac{\partial W}{\partial \mathbf{p}_i}. \tag{3.13c}$$

Equation (3.13a) is very important, because it relates the global systems dynamics to the local systems dynamics and provides the basis for a direct approach to the quantum many-body theory. The use of a many-times approach is not new and dates back to Dirac *et al* (1932). Our approach is close to that of Longhi *et al* (1986). As will be shown in a subsequent publication, equation (3.13a) makes it possible to relate the individual quantum systems to the global quantum system using one (universal) wave function.

3.7. Global System Properties

Theorem 3.5. The dynamical laws of physics formulated using the canonical proper-time implementation will be covariant for all observers.

Theorem 3.6. There is a many-particle direct-interaction theory with the following properties:

1. The theory satisfies the first two postulates of special relativity.
2. The theory is based on Hamiltonian dynamics.
3. The theory is based on independent (canonical) particle variables.

It is known that replacement of the first condition with the requirement of Lorentz covariance is only compatible with noninteracting particles. This is the content of the no-interaction theorem (see Currie *et al*, 1963, and references therein).

In the study of physical systems, one is interested in either the behavior of the entire system or some subsystem. Experimental measurements are made in this manner. The cluster decomposition property is a requirement of any theory purporting to be a possible representation of the real world. Basically this is the property that, if any two or more subsystems become widely separated, then they may be treated as independent systems (clusters).

Theorem 3.7. Let the above system be decomposed into two or more clusters. Then there exists a (unique) proper-clock and canonical Hamiltonian for each cluster.

Proof. We assume that the two subsystems are sufficiently separated that all observers can agree that they are distinct. In this case, each observer can identify masses M_1 and M_2 along with Hamiltonians H_1 and H_2. It follows that $d\tau_1 = (M_1c^2)/H_1dt$ and $d\tau_2 = (M_2c^2)/H_2dt$, so that each observer can construct a (proper-time) relativistic theory for each cluster.

Theorem 3.7 is true without the assumption that the subsystems interact weakly, so that we can consider collective systems based on the detailed interactions of individual particles. This is less difficult than the various phenomenological approaches, which require both model justification and consistency analysis prior to use.

The following theorem tells us that, although there is no unique rest frame for the universe, under very mild conditions it does have a unique clock and Hamiltonian which is available to all observers.

Theorem 3.8. Assume that the universe has finite mass and energy, and that each observer can choose a local inertial frame from which his region of the universe is at rest relative to the observed system. Then there exists a unique proper-clock and Hamiltonian for the universe.

Proof. Applying the cluster decomposition theorem, our observer can identify masses M_1 for his region of the universe and M_2 for the complement region, along with Hamiltonians H_1 and H_2. It follows that $H = H_1 + H_2$, $M = M_1 + M_2$, and $d\tau = (Mc^2/H)dt$ define the global mass and Hamiltonian for the universe. We can now construct our proper-time Hamiltonian K. Since M and H are fixed and invariant for all observers, we see that both K and τ are unique and invariant for all observers. (Note that the M_i and H_i will vary with the observer, reflecting the nonuniqueness of inertial frames.) This proof does not strictly adhere to our method and minor modifications are required if we assume the masses depend on the particle variables (isotopes). The final result will be the same.

Theorem 3.9. If the universe is not finite and does not have a finite amount of energy, but we assume that the observable universe is representable in the sense that the observed ratio of mass to energy is constant and independent of our observed portion of the universe, then the universe has a unique clock.

Proof. The proof follows since $d\tau = (Mc^2/H)dt$ will be independent of the representable portion of the universe.

It has been known for some time (Rowan-Robinson, 1996) that the cosmic background radiation provides a frame in which the universe looks isotropic. Furthermore, this frame is available to all observers.

4. LOCAL PARTICLE WORLD VIEW

4.1. Particle View

We now inquire as to the world view as seen by the ith particle in a local sense. In this case, the particle is viewed as responding to a given field with potentials $\mathbf{A}_i$ and V_i. To obtain this view, we transform from the observer clock to the proper-clock of the ith particle along with its proper-time Hamiltonian. From (3.13b), we have

$$K_i = \frac{\boldsymbol{\pi}_i^2}{2m_i} + m_i c^2 + V_i \frac{H_{0i}}{m_i c^2} + \frac{V_i^2}{2m_i c^2}, \tag{4.1a}$$

so from (3.13c) we have:

$$\mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau_i} = \sum_j \frac{\partial K_i}{\partial \mathbf{p}_j} T_i \delta_{ij} = T_i \frac{\partial K_i}{\partial \mathbf{p}_i} = \frac{\boldsymbol{\pi}_i}{m_i} \Rightarrow \mathbf{p}_i = m_i \mathbf{u}_i + \frac{e_i}{c} \mathbf{A}_i. \tag{4.1b}$$

Using standard elementary methods and notation, we get

$$\begin{aligned} \frac{d\mathbf{p}_i}{d\tau_i} &= -\sum_j \frac{\partial K_i}{\partial \mathbf{x}_j} T_i \delta_{ij} \\ &= -T_i \frac{\partial K_i}{\partial \mathbf{x}_i} = \frac{e_i}{c}\frac{d\mathbf{A}_i}{d\tau_i} - \frac{e_i}{c}\frac{\partial \mathbf{A}_i}{\partial \tau_i} - \nabla_i(V_i)\frac{H_{0i}}{m_i c^2} + \frac{e_i}{c}\mathbf{u}_i \times \mathbf{B}_i, \end{aligned} \tag{4.2a}$$

or, using $H_{0i}/m_i c^2 = \sqrt{1 + u^2/c^2} = b_i/c$, we have

$$\frac{m_i c}{b_i}\frac{d\mathbf{u}_i}{d\tau_i} = \left\{ e_i \mathbf{E}_i + \frac{e_i}{b_i}\mathbf{u}_i \times \mathbf{B}_i \right\} = \mathbf{F}_i, \qquad \mathbf{E}_i = -\frac{1}{b_i}\frac{\partial \mathbf{A}_i}{\partial \tau_i} - \nabla_i \Phi_i, \tag{4.2b}$$

We thus get a proper-time version of the Lorentz force. This is the force a local observer would obtain using the proper-clock of the particle. This equation is nonlinear in u_i (because of the b_i terms). It may cause some concern that we have derived an equation that uses the proper-time and yet is different from the one we get using the standard approach (which parametrizes with the particle proper-time variable). Using $(1/b_i)(\partial/\partial\tau_i) = (1/c)(\partial/\partial t)$ and $\mathbf{w}_i/c = \mathbf{u}_i/b_i$ (where $\mathbf{w}_i = d\mathbf{x}_i/dt$), we can write (4.2b) as:

$$m_i \frac{d\mathbf{u}_i}{dt} = \left\{ e_i \mathbf{E}_i + \frac{e_i}{c} \mathbf{w}_i \times \mathbf{B}_i \right\}, \qquad \mathbf{E}_i = -\frac{1}{c} \frac{\partial \mathbf{A}_i}{\partial t} - \nabla_i \Phi_i. \tag{4.2c}$$

It follows that (4.2c) is completely equivalent to (4.2b) mathematically. *They are clearly not physically equivalent since the neighborhood of the particle is curved due to the interaction of the particle field with the external field.*

Equations (4.2c) and (4.2b) do not include a dissipation term which accounts for the observed radiation of accelerated charges. This radiation is known to occur instantaneously with acceleration and its nature has been the object of much speculation (Wheeler and Feynman, 1945). In the standard approach, Maxwell's equations (for the fields of the *i*th particle) are used to compute the energy radiated from a tube along the world line of the particle. The negative of the computed term is added to the Lorentz force [the right-hand side of (4.2c)] to provide the appropriate dissipation term, which leads to the Lorentz–Dirac equation. This approach was first used by Dirac (1938). See Rohrlich (1965) for a comprehensive overview of the classical theory up to that time including a complete review of the history of the subject. Rohrlich (1997) provides a more recent review which is highly enlightening for those unaware of the continuing effort to solve the classical electron problem. Another recent critical review of the problems can be found in de Souza (1987). See Yaghjian (1992), Parrott and Endres (1995), and Trump and Schieve (1997) for related issues.

Finally, note that in computing (4.1b) and (4.2a) we have used the canonical variables for all the particles in the system (universe) and yet they have no impact on the expressions for the velocity and force on the *i*th particle. In this sense, they may be considered "hidden variables." In Section 5, we will return to this problem and show that, on the global level, they are not hidden.

4.2. Field View

There have been a number of attempts to generalize either Maxwell's equations, the Lorentz force, or both. Ritz (1908a) wanted to eliminate the fields completely and replace them by "elementary (retarded) actions." Driscall (1992, 1997a,b) has recently proposed a revised version of the Ritz theory. Wheeler and Feynman (1949) revived the action-at-a-distance theory of Schwarzschild (1903), Tetrode (1922), and Fokker (1929a,b, 1932) in order to construct a theory that could lead to a better understanding of the problems at the quantum level. This approach had a direct impact on Feynman's later work on QED (Feynman, 1948). That Feynman was looking for a generalization of classical electrodynamics is witnessed by a paper of Dyson (1990). In 1948, Feynman showed Dyson that the Lorentz force and the homogeneous Maxwell equations could be obtained using the canonical commutation relations between the vector parts of the coordinates and velocities. Motivated by the Dyson paper, Tanimura (1992) constructed a Lorentz-covariant generalization of Feynman's approach. It was shown by Land *et al* (1995) that, within the framework of the (five-dimensional) proper-time method, Tanimura's theory does not lead to the usual Maxwell equations for a number of important reasons. When corrected, his theory corresponds to that of Saad *et al* (1989). Their theory is derived from the Horwitz–Piron (1973) Stueckelberg-type quantum theory. Dirac (1951a,b) also returned to the classical electron problem and was never completely satisfied with the current formulation.

In our approach, we have replaced t by τ and acquired K as its canonical Hamiltonian, so that τ becomes both our coordinate time and evolution parameter. It was noted by Santilli (private communication) that this is equivalent to imposing a constraint on the Horwitz–Piron clock and has the same effect on the corresponding group theory of Aghassi *et al* (1970a,b).

In order to write Maxwell's equations using the proper-time of the source, we began with the following theorem, which follows directly from the proper-time group [equations (1.6a) and (1.6b)].

Theorem 4.1. If we set $b^2 = (u^2 + c^2)$, then we have (1)

$$\frac{\mathbf{w}}{c} = \frac{\mathbf{u}}{b}, \qquad \nabla_{\mathrm{X}} = \gamma(v)[\nabla_{\mathrm{X}'} - (\mathbf{v}/c^2)\delta(u')^{-1}\partial_\tau], \tag{4.3a}$$

$$\frac{1}{c}\partial_t = \frac{1}{b}\partial_\tau, \qquad \partial_t = \gamma(v)(\delta(u')^{-1}\partial_\tau - \mathbf{v}\cdot\nabla_{\mathrm{X}'}); \tag{4.3b}$$

and (2) Maxwell's equations as seen from X for the field of the source are

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho,$$

$$\nabla\times\mathbf{E} = -\frac{1}{b}\frac{\partial\mathbf{B}}{\partial\tau}, \qquad \nabla\times\mathbf{B} = \frac{1}{b}\left[\frac{\partial\mathbf{E}}{\partial\tau} + 4\pi\rho\mathbf{u}\right]. \tag{4.4}$$

We see that the velocity of electromagnetic waves with respect to τ *depends on the motion of the source, and their magnitude is always larger than c (but less than b).* This observation may seem strange and even contradictory to the second postulate, but it is not. On closer inspection, we realize that the second postulate refers to the observer's point of view using his measuring rods and clock. There is no contradiction since we are using the observer's measuring rods and the *clock of the source. The important point is that the dependence of the speed of light on the motion of the source reflects a choice of conventions in formulating physical theory.* We will discuss this point later.

In the Michelson–Morley experiment, the source is at rest in the frame of the observer so that $u = 0$ and $b = c$. It follows that our approach explains the Michelson–Morley null result. It also provides agreement with the conceptual (but not technical) framework proposed by Ritz (1908a); namely, that the speed of light depends on the (proper) motion of the source. In this sense, both Einstein and Ritz were correct. The next result was proven by Gill *et al* (1997). However, it follows from the same calculations as in Einstein (1905), using (4.3a) and (4.3b).

Theorem 4.2. Maxwell's equations are covariant under the proper-time group.

Let us now inquire as to the nature of the fields which act on the *i*th particle to cause its motion. For the action of the *j*th particle on the *i*th particle, equations (4.4) become

$$\nabla_j \cdot \mathbf{B}_{ij} = 0, \qquad \nabla_j \cdot \mathbf{E}_{ij} = 4\pi\rho_j,$$
$$\nabla_j \times \mathbf{E}_{ij} = -\frac{1}{b_j}\frac{\partial \mathbf{B}_{ij}}{\partial \tau_j}, \nabla_j \times \mathbf{B}_{ij} = \frac{1}{b_j}\left[\frac{\partial \mathbf{E}_{ij}}{\partial \tau_j} + 4\pi\rho_j \mathbf{u}_j\right]. \tag{4.5}$$

In what follows, we will suppress the indices until the end of the section, with the understanding that they refer to the action of the *j*th particle on the *i*th particle. Returning to (4.5), we perform the standard manipulations, using

$$\mathbf{E} = -\frac{1}{b}\frac{\partial \mathbf{A}}{\partial \tau} - \nabla\Phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{4.6}$$

to obtain

$$\nabla\left[\nabla \cdot \mathbf{A} + \frac{1}{b}\frac{\partial \Phi}{\partial \tau}\right] + \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{1}{b}\frac{\partial \mathbf{A}}{\partial \tau}\right] - \nabla^2 \mathbf{A} = \frac{1}{b}(4\pi \mathbf{J}), \tag{4.7}$$

and

$$-\nabla^2\Phi - \frac{1}{b}\frac{\partial}{\partial \tau}[\nabla \cdot \mathbf{A}] = 4\pi\rho. \tag{4.8}$$

Imposing the (proper-time) Lorentz gauge

$$\nabla \cdot \mathbf{A} + \frac{1}{b}\frac{\partial \Phi}{\partial \tau} = 0, \tag{4.9}$$

we get the wave equations

$$\frac{1}{b}\frac{\partial}{\partial\tau}\left[\frac{1}{b}\frac{\partial \mathbf{A}}{\partial\tau}\right] - \nabla^2\mathbf{A} = \frac{1}{b}(4\pi\mathbf{J}), \tag{4.10a}$$

$$\frac{1}{b}\frac{\partial}{\partial\tau}\left[\frac{1}{b}\frac{\partial \Phi}{\partial\tau}\right] - \nabla^2\Phi = 4\pi\rho. \tag{4.10b}$$

Straightforward calculations with (4.10b) lead to an equation of the form

$$-\nabla^2\Phi - \frac{1}{b^4}\left[\mathbf{u}\cdot\frac{d\mathbf{u}}{d\tau}\right]\frac{\partial\Phi}{\partial\tau} + \frac{1}{b^2}\frac{\partial^2\Phi}{\partial\tau^2} = 4\pi\rho. \tag{4.11a}$$

We get a similar equation for **A**.

We get the same form for the equations if we derive them directly without using potentials (see Section 5.2). The second term on the left-hand side of equation (4.11) is a dissipative part of the wave equation. It is zero if **u** is constant and arises instantaneously with acceleration. This is what we expect of a radiation reaction (Wheeler and Feynman, 1945). It is also of interest to observe that we have made no assumptions about the structure of the source. It is easy to see that the dissipative part of (4.11a) can be written using the observer's clock as

$$-\frac{1}{b^4}\left[\mathbf{u}\cdot\frac{d\mathbf{u}}{d\tau}\right] = -\frac{\mathbf{w}_i\cdot\dot{\mathbf{w}}_i}{\sqrt{1-(\mathbf{w}_i/c)^2}}. \tag{4.11b}$$

In order to solve (4.11a), the simplest assumption is that $a^2 = -1/b^2[\mathbf{u}\cdot d\mathbf{u}/d\tau]$ and $1/b^2$ may be treated as constants (take mean square averages). We can then obtain the Green's function from:

$$-\nabla^2\mathbf{G} + \frac{a^2}{b^2}\frac{\partial\mathbf{G}}{\partial\tau} + \frac{1}{b^2}\frac{\partial^2\mathbf{G}}{\partial\tau^2} = 4\pi\delta(\mathbf{r}-\mathbf{r}_0)\delta(\tau-\tau_0). \tag{4.12a}$$

If $R = |\mathbf{r} - \mathbf{r}_0|$ and $t = \tau - \tau_0$, the solution (Morse and Feshbach, 1953, p. 868) is

$$\mathbf{G}(R, t) = \mathbf{G}_1(R, t) + \mathbf{G}_2(R, t),$$

$$\mathbf{G}_1(R, t) = 1/R\exp\left\{-\frac{1}{2}a^2t\right\}\delta(t - R/b), \tag{4.12b}$$

$$\mathbf{G}_2(R, t) = -a^2[2bh(R, t)]^{-1}\exp\left\{-\frac{1}{2}a^2t\right\}\mathbf{I}_1\left[\frac{1}{2}a^2h(R, t)\right]\mathbf{U}(bt - R),$$

with

$$h(R, t) = (t^2 - R^2/b^2)^{1/2}, \qquad \mathbf{U}(x) = \begin{cases} = 0, & x < 0 \\ = 1, & x \geq 0 \end{cases} \tag{4.12c}$$

representing the step function, and $\mathbf{I}_1$ is a modified Bessel function, $\mathbf{I}_1(x) = -i\mathbf{J}_1(ix)$.

Both terms arise because of our use of the proper-clock and will never appear in the frame of a comoving observer. The first term is Yukawa-like and shows that acceleration creates an effective mass for the field of the source. A number of writers (de Broglie, 1940; Schrödinger and Bass, 1955; Costa de Beauregard (1997a,b) have called into question the (identically) zero nature of the photon mass.

The second term may be viewed as a wake that moves out from the particle and represents an inertial drag created by the particle's resistance to acceleration. Note that both terms in (4.12b) carry intrinsic information about the source, so that if the wave is split (as for example in the Aspect *et al* (1982) experiment, the two resulting waves will always be correlated.

In order to gain further insight, set $\Phi = (b/c)^{1/2}\mathbf{g}$ so that our equation is transformed to

$$\frac{1}{b^2}\frac{\partial^2 \mathbf{g}}{\partial \tau^2} - \nabla^2 \mathbf{g} + \left[\frac{\ddot{b}}{2b^3} - \frac{5\dot{b}^2}{4b^4}\right]\mathbf{g} = 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(\tau - \tau_0). \tag{4.13a}$$

We can write the last term on the left-hand side of (4.12) as:

$$\mu^2 = \left[\frac{\ddot{b}}{2b^3} - \frac{5\dot{b}^2}{4b^4}\right] = \left[\frac{(\dot{\mathbf{u}})^2}{2b^4} + \frac{\mathbf{u}\cdot\ddot{\mathbf{u}}}{2b^4} - \frac{5(\mathbf{u}\cdot\dot{\mathbf{u}})^2}{4b^6}\right]. \tag{4.13b}$$

Independently of the resemblance of (4.13a) to the Klein–Gordon equation, classical field equations of this type arise in the study of a flexible string embedded in a thin rubber sheet with additional stiffness forces coming from the rubber. In this case the string experiences another restoring force, caused by the rubber, along its length. By analogy then, we may interpret the field equations as providing a "virtual medium" which acts instantaneously to resist attempts by the particle to accelerate. *This virtual medium (the particle's self-field) could certainly be called the source of the (electromagnetic) inertial reaction to attempts to accelerate the particle.*

Let us assume that μ^2 and $1/b$ may be (approximately) treated as constants. Then, as before, we have $\mathbf{g}(R, t) = \mathbf{g}_1(R, t) + \mathbf{g}_2(R, t)$, with

$$\mathbf{g}_1(R, t) = \frac{1}{R}\,\delta\left(t - \left(\frac{R}{b}\right)\right),$$

$$\mathbf{g}_2(R, t) = \frac{-\mu}{\sqrt{t^2 - \left(\frac{R}{b}\right)^2}}\,\mathbf{L}_1\left[\mu b\sqrt{t^2 - \left(\frac{R}{b}\right)^2}\right]\mathbf{U}(bt - R), \quad (4.14)$$

$$\mathbf{U}(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}, \qquad R = |\mathbf{r} - \mathbf{r}_0|, \quad t = \tau - \tau_0,$$

and $\mathbf{L}_1 = \mathbf{J}_1$ if $\mu^2 > 0$ and $\mathbf{L}_1 = \mathbf{I}_1$ if $\mu^2 < 0$. It is important to note that $\mathbf{L}_1(z)/z$ has a constant value at $z = 0$. Thus, at $\tau - \tau_0 = (R/b)^+$, $\mathbf{g}_2$ has the value $-a\mu$, where a is a constant (the plus sign means that $\tau - \tau_0$ approaches R/b from the right). Since we have used an approximation to solve equation (4.12), we cannot go too far with the physical interpretation. However, the above argument seems to indicate strongly that the effective mass part travels outward in a radial direction and can be viewed as the photon part of the field. This effective mass will increase with acceleration of the source, so that this term will act more and more like a particle. This term provides precisely the behavior predicted by Ritz (1980a,b) (see also Ehrenfest, 1912).

We can integrate to get that ($R = |\mathbf{r} - \mathbf{r}_0|$, $t = \tau - \tau_0$):

$$\Phi(\mathbf{r}, \tau) = \frac{1}{4\pi}\int_0^\tau \left(\frac{b}{c}\right)^{1/2} d\tau_0 \int_{R^3} d\mathbf{r}_0[\mathbf{g}_1(R, t) + \mathbf{g}_2(R, t)]\rho(\mathbf{r}_0, \tau_0)$$

$$= \Phi_1 + \Phi_2, \quad (4.15a)$$

$$\mathbf{A} = (\mathbf{u}/b)[\Phi_1(\mathbf{r}, \tau) + \Phi_2(\mathbf{r}, \tau)]; \quad (4.15b)$$

where, as before, $t = \tau - \tau_0$. Reverting back to our indices, we can write the equations for the potentials which generate the total fields caused by other particles (acting on the ith particle) as $\Phi_i(\mathbf{r}_i, \tau) = \Phi_{i1} + \Phi_{i2}$,

$$\Phi_{i1} = \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{ij1}(\mathbf{r}_{ij}, \tau_i), \qquad \Phi_{i2} = \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{ij2}(\mathbf{r}_{ij}, \tau_i), \quad (4.16a)$$

$$\mathbf{A}_{ij}(r_{ij}, \tau_i) = \frac{\mathbf{u}_i}{b_i}[\Phi_{ij1}(r_{ij}, \tau_i) + \Phi_{ij2}(r_{ij}, \tau_i)]. \quad (4.16b)$$

We must still compute the individual fields from each particle to get the total field which causes the ith particle to accelerate. That is,

$$\mathbf{E}_i = -\frac{1}{b_i}\frac{\partial}{\partial \tau_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{A}_{ij}(r_{ij}, \tau_i)\right) - \boldsymbol{\nabla}_i(\Phi_i), \tag{4.17a}$$

$$\mathbf{B}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \nabla_i \times \mathbf{A}_{ij}(r_{ij}, \tau_i). \tag{4.17b}$$

To the above equations we must add the Lorentz force (4.2c) to get the complete picture from the local point of view. In closing this section, we recognize that our results are based on qualitative analysis and hence do not provide quantitative information. We will consider particular cases at a later time after a complete analysis of the Green's function corresponding to (4.10) has been finished. The most important points are that, in general, the field equations are dissipative and they carry intrinsic information about the velocity and acceleration of the source.

5. GLOBAL PARTICLE WORLD VIEW

5.1. Particle View

We now look at the dynamics of the *i*th particle from the global point of view. This means that we use the individual particle variables and the proper-time of the global system. It is this view that represents new physics and helps us to understand how the global system controls the local dynamics.

Theorem 5.1. Assuming that K_i has no explicit dependence on τ_i, we have

$$dK_i/d\tau_i = 0, \qquad dK/d\tau \neq 0. \tag{5.1a}$$

Proof. It is easy to see that $dK_i/d\tau_i = 0$. From (3.6), we have

$$\mathbf{v}_i = \frac{d\mathbf{x}_i}{d\tau} = T\frac{\partial K_i}{\partial \mathbf{p}_i} = \frac{d\tau_i}{d\tau}T_i\frac{\partial K_i}{\partial \mathbf{p}_i} = \frac{d\tau_i}{d\tau}\frac{\boldsymbol{\pi}_i}{m_i} \Rightarrow \tag{5.1b}$$

$$\frac{dK_i}{d\tau} = \sum_{i=1}^{n} (\mathbf{v}_j\cdot\boldsymbol{\nabla}_j)K_i + T_i^{-1}\frac{d\mathbf{p}_i}{d\tau}\cdot\mathbf{u}_i \neq 0. \tag{5.1c}$$

It follows that, from the global point of view, the ith particle does not conserve energy. For the force we have

$$\frac{d\mathbf{p}_i}{d\tau} = -T\frac{\partial K}{\partial \mathbf{x}_i} = -\sum_{j=1}^{n}\frac{d\tau_j}{d\tau}T_j\frac{\partial K_j}{\partial \mathbf{x}_i}$$

$$= \sum_{j=1}^{n}\frac{d\tau_j}{d\tau}\left\{\frac{e_j}{c}[(\mathbf{u}_j\cdot\boldsymbol{\nabla}_i)\mathbf{A}_j + \mathbf{u}_j\times(\boldsymbol{\nabla}_i\times\mathbf{A}_j)] - \frac{b_j}{c}\boldsymbol{\nabla}_i(V_j)\right\}. \tag{5.2}$$

From $cdt = b_i d\tau_i$ and $cdt = bd\tau$ we get that

$$\frac{d\tau_i}{d\tau} = \frac{b}{b_i}. \tag{5.3}$$

Using (3.4), (3.13), and (5.3), assuming that there are no (global) external fields, we have that

$$\mathbf{v}_i = \frac{b}{b_i}\,\mathbf{u}_i, \tag{5.4a}$$

$$\boldsymbol{\nabla}_i\mathbf{A}_j = -\boldsymbol{\nabla}_j\mathbf{A}_{ji}, \qquad \boldsymbol{\nabla}_i\mathbf{A}_i = \sum_{j\neq i}^{n} \boldsymbol{\nabla}_i\mathbf{A}_{ij}, \tag{5.4b}$$

$$\boldsymbol{\nabla}_i V_j = -\boldsymbol{\nabla}_j V_{ji}, \qquad \boldsymbol{\nabla}_i V_i = \sum_{j\neq i}^{n} \boldsymbol{\nabla}_i V_{ij}. \tag{5.4c}$$

Setting $\mathbf{B}_{ij} = \boldsymbol{\nabla}_i \times \mathbf{A}_{ij}$ and using

$$(\mathbf{v}_j\cdot\boldsymbol{\nabla}_j)\mathbf{A}_{ji} = \frac{d\mathbf{A}_{ji}}{d\tau} - \frac{\partial\mathbf{A}_{ji}}{\partial\tau}, \tag{5.4d}$$

we can write (5.2) as

$$\frac{c}{b}\frac{d\mathbf{p}_i}{d\tau} = \sum_{j\neq i}^{n}\left(\frac{e_i}{b}\frac{d\mathbf{A}_{ij}}{d\tau} - \frac{e_j}{b}\frac{d\mathbf{A}_{ji}}{d\tau}\right) + \sum_{j\neq i}^{n} (\mathbf{F}_{ij} - \mathbf{F}_{ji}), \tag{5.5a}$$

where $\mathbf{E}_{ij} = -\,(1/b)d\mathbf{A}_{ij}/d\tau - \boldsymbol{\nabla}_i(\Phi_i)$ and $\mathbf{F}_{ij} = e_i\mathbf{E}_{ij} + (e_i/b)\mathbf{v}_i \times \mathbf{B}_{ij}$.

It follows from (5.5a) that

$$\frac{d\mathbf{P}}{d\tau} = \sum_{i=1}^{n} \frac{d\mathbf{p}_i}{d\tau} = 0, \tag{5.5b}$$

so that we automatically get conservation of momentum. Note that:

$$\sum_{j\neq i}^{n} \frac{e_i}{b}\frac{d\mathbf{A}_{ij}}{d\tau} + \sum_{j\neq i}^{n} \mathbf{F}_{ij} = \frac{e_i}{b}\frac{d\mathbf{A}_i}{d\tau} + \mathbf{F}_i), \tag{5.6a}$$

so that we can write (5.5a) as

$$\frac{c}{b}\frac{d\mathbf{p}_i}{d\tau} = \frac{e_i}{b}\frac{d\mathbf{A}_i}{d\tau} + \mathbf{F}_i - \sum_{j\neq i}^{n}\left(\frac{e_j}{b}\frac{d\mathbf{A}_{ji}}{d\tau} + \mathbf{F}_{ji}\right). \tag{5.6b}$$

Since $\mathbf{p}_i = m_i\mathbf{u}_i + e_i/c\mathbf{A}_i$, we also see that

$$\frac{m_i c}{b}\frac{d\mathbf{u}_i}{d\tau} = \mathbf{F}_i - \sum_{j \neq i}^{n} \left\{ \mathbf{F}_{ji} + \frac{e_i}{b}\frac{d\mathbf{A}_{ji}}{d\tau} \right\}, \tag{5.7}$$

$$\mathbf{F}_i = \left\{ e_i \mathbf{E}_i + \frac{e_i}{b}\mathbf{v}_i \times \mathbf{B}_i \right\}, \qquad \mathbf{E}_i = -\frac{1}{b}\frac{\partial \mathbf{A}_i}{\partial \tau} - \nabla_i(\Phi_i), \tag{5.8}$$

where now

$$\Phi_i(\mathbf{r}_i, \tau) = \Phi_{i1} + \Phi_{i2}, \tag{5.9}$$

$$\Phi_{i1} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_{ij1}(\mathbf{r}_{ij}, \tau), \qquad \Phi_{i2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_{ij2}(\mathbf{r}_{ij}, \tau), \tag{5.10}$$

$$\mathbf{A}_{ij}(r_{ij}, \tau) = \frac{\mathbf{v}_i}{b}[\Phi_{ij1}(r_{ij}, \tau) + \Phi_{ij2}(r_{ij}, \tau)]. \tag{5.11}$$

Comparing with (4.2c),

$$\frac{m_i c}{b_i}\frac{d\mathbf{u}_i}{d\tau_i} = \left\{ e_i\, \mathbf{E}_i + \frac{e_i}{b_i}\mathbf{u}_i \times \mathbf{B}_i \right\} = \mathbf{F}_i, \qquad \mathbf{E}_i = -\frac{1}{b_i}\frac{\partial \mathbf{A}_i}{\partial \tau_i} - \nabla_i(\Phi_i),$$

we see that the force $\mathbf{F}_i$ in (5.7) is of the same form as in (4.2c) with $(1/b_i)(\partial/\partial\tau_i)$ replaced by $(1/b)(\partial/\partial\tau)$. Since $(1/b_i)(\partial/\partial\tau_i) = (1/b)(\partial/\partial\tau)$ and $\mathbf{u}_i/b_i = \mathbf{v}_i/b$, they are equal. With equation (5.7), we see that the addtional terms are the long-sought reaction force of the ith particle on all other particles in the system (Newton's third law). It is these terms which carry away the energy of radiation caused by the external forces that act on the ith particle. Recall that these terms are functions of the velocity and acceleration of the ith particle.

The following physical picture arises. When the proper-clock of the particle is used, we get the correct fields because they are local effects. However, when we compute the force equation, we only get the local part. When we compute the force equation using the proper-clock for the system, we obtain the long-sought dissipative term directly.

Before finishing this section, let us note that we have used $\mathbf{u}_i$ on the left-hand side of (5.7) so that a clear comparison can be made. In order to see the full impact of the global system on the local dynamics, we use $\mathbf{u}_i/b_i = \mathbf{v}_i/b$ to solve for $\mathbf{u}_i$ in terms of $\mathbf{v}_i$, so that

$$\mathbf{u}_i = \frac{c\mathbf{v}_i}{\sqrt{b^2 - \mathbf{v}_i^2}}, \qquad b_i = \frac{cb}{\sqrt{b^2 - \mathbf{v}_i^2}}, \qquad \frac{b_i}{b} = \frac{c}{\sqrt{b^2 - \mathbf{v}_i^2}}. \tag{5.12}$$

From (3.10c) we know that $b^2 = \mathbf{U}^2 + c^2$, so that

$$m_i \frac{d\mathbf{u}_i}{d\tau} = \frac{m_i c}{\sqrt{b^2 - \mathbf{v}_i^2}} \frac{d\mathbf{v}_i}{d\tau} - \frac{m_i c \mathbf{v}_i}{\sqrt{b^2 - \mathbf{v}_i^2}} \left\{ \mathbf{U} \bullet \frac{d\mathbf{U}}{d\tau} - \mathbf{v}_i \bullet \frac{d\mathbf{v}_i}{d\tau} \right\}. \tag{5.13a}$$

Combining (5.7b) and (5.13a), we have:

$$\frac{c}{b} \frac{m_i c}{\sqrt{b^2 - \mathbf{v}_i^2}} \frac{d\mathbf{v}_i}{d\tau} - \frac{c}{b} \frac{m_i c \mathbf{v}_i}{\sqrt{b^2 - \mathbf{v}_i^2}} \left\{ \mathbf{U} \bullet \frac{d\mathbf{U}}{d\tau} - \mathbf{v}_i \cdot \frac{d\mathbf{v}_i}{d\tau} \right\}$$

$$= \mathbf{F}_i - \sum_{j \neq i}^{n} \left\{ \mathbf{F}_{ji} + \frac{e_j}{b} \frac{d\mathbf{A}_{ji}}{d\tau} \right\}. \tag{5.13b}$$

Note that from (5.12), the fact that $b^2 = \mathrm{U}^2 + c^2$, and since $b > c$, that $\mathbf{v}_i$ can be larger than c. If $dM/d\tau$ and $d\mathbf{U}/d\tau$ are both zero, this still does not imply that $\mathbf{U}$ is zero, so that it is still possible for the velocity of a particle to be faster than the speed of light (relative to the source). Assuming $\mathbf{U}$ is zero implies that $b = c$, so our equations become ($\beta_i = \mathbf{v}_i/c$)

$$\frac{m_i}{\sqrt{1 - \beta_i^2}} \frac{d\mathbf{v}_i}{d\tau} + \frac{m_i}{\sqrt[3]{1 - \beta_i^2}} \left\{ \mathbf{v}_i \bullet \frac{d\mathbf{v}_i}{d\tau} \right\} \mathbf{v}_i$$

$$= \mathbf{F}_i - \sum_{j \neq 1}^{n} \left\{ \mathbf{F}_{ji} + \frac{e_j}{c} \frac{d\mathbf{A}_{ji}}{d\tau} \right\}. \tag{5.13c}$$

In this case, the global system appears at rest in the frame of reference of our observer. Even here, we get a theory that is distinct from the standard approach in that we have the dissipation term without any additional work. Assume that $d\mathbf{A}_{ji}/d\tau = 0$ and notice that, in the general case, the work done by (5.13b) can be written as [using (5.12)]:

$$\frac{1}{2} m_i c^2 \sqrt{1 - \mathbf{v}_i^2/b^2} \frac{d}{d\tau} \left(\frac{\mathbf{v}_i}{\sqrt{b^2 - \mathbf{v}_i^2}} \right)^2 = \left\{ \mathbf{v}_i \bullet \mathbf{F}_i - \sum_{j \neq 1}^{n} \mathbf{v}_i \bullet \mathbf{F}_{ji} \right\}. \tag{5.13d}$$

Although $\mathbf{v}_i$ is orthogonal to $\mathbf{v}_i \times \mathbf{B}_i$, it is not orthogonal to $\mathbf{v}_j \times \mathbf{B}_{ji}$ so the above equation becomes:

$$\frac{1}{2} m_i c^2 \sqrt{1 - \mathbf{v}_i^2/b^2} \frac{d}{d\tau} \left(\frac{\mathbf{v}_i}{\sqrt{b^2 - \mathbf{v}_i^2}} \right)^2$$

$$= e_i \mathbf{v}_i \bullet \mathbf{E}_i - \sum_{j \neq i}^{n} \{ e_j \mathbf{v}_i \bullet \mathbf{E}_{ji} + (e_j \mathbf{v}_i/b) \bullet \mathbf{v}_j \times \mathbf{B}_{ji} \}. \tag{5.13e}$$

The left-hand side of (5.13e) is the effective kinetic energy of the ith particle. It follows that the energy loss due to radiation is a very complicated process and can only be understood in simple cases (assuming exact solutions for the field equations).

5.2. Global Field View

It should be noted that equations (5.8)–(5.11) were obtained by replacing $\mathbf{u}_i/b_i$ by $\mathbf{v}_i/b$. This reflects the fact that radiation is a local effect and must be computed using the proper-time of the particle involved. On the other hand, a true picture of the particle dynamics requires the use of the global proper-time. It follows that the complete picture requires both the individual and global proper-time variables. Furthermore, these are the only unique variables intrinsic to the system and available to all observers.

Let us now derive the field equations by directly using the global proper-time variable. Using $1/c\partial_t = 1/b\partial_\tau$, we write Maxwell's equations at any point in the domain:

$$\nabla \cdot \mathbf{B} = 0, \ \nabla \cdot \mathbf{E} = 4\pi\rho, \qquad\qquad \nabla \times \mathbf{E} = -\frac{1}{b}\frac{\partial \mathbf{B}}{\partial \tau}, \qquad \nabla \times \mathbf{B} = \frac{1}{b}\left[\frac{\partial \mathbf{E}}{\partial \tau} + 4\pi \mathbf{J}\right], \tag{5.14}$$

where now ρ and $\mathbf{J}$ represent the charge and current density at the site of concern. Using standard identities and noting that b depends on $\mathbf{U}$, these equations lead to

$$\frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{1}{b}\frac{\partial \mathbf{E}}{\partial \tau}\right] - \nabla^2 \mathbf{E} = -\nabla(4\pi\rho) - \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{4\pi \mathbf{J}}{b}\right], \qquad \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{1}{b}\frac{\partial \mathbf{B}}{\partial \tau}\right] - \nabla^2 \mathbf{B} = \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{4\pi\nabla \times \mathbf{J}}{b}\right]. \tag{5.15}$$

This last equation can also be written as

$$\frac{1}{b^2}\frac{\partial^2 \mathbf{E}}{\partial \tau^2} - \left(\frac{\mathbf{U}}{b^4} \cdot \frac{d\mathbf{U}}{d\tau}\right)\frac{\partial \mathbf{E}}{\partial \tau} - \nabla^2 \mathbf{E} = -\nabla(4\pi\rho) - \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{4\pi \mathbf{J}}{b}\right], \tag{5.16}$$

$$\frac{1}{b^2}\frac{\partial^2 \mathbf{B}}{\partial \tau^2} - \left(\frac{\mathbf{U}}{b^4} \cdot \frac{d\mathbf{U}}{d\tau}\right)\frac{\partial \mathbf{B}}{\partial \tau} - \nabla^2 \mathbf{B} = \frac{1}{b}\frac{\partial}{\partial \tau}\left[\frac{4\pi\nabla \times \mathbf{J}}{b}\right]. \tag{5.17}$$

From (5.16) and (5.17) we see that, from the global point of view, the fields dissipate energy (radiation) at every point in the domain of the system. Since $\mathbf{U} = 1/M \sum m_j \mathbf{u}_j$, this radiation depends on the average of the (proper) motion of all the particles in the system. As this term includes $\mathbf{u}_i$ for each i, all particles in the system will experience a self-interaction as part of this average.

The radiation will appear thoughout the domain of the system and this implies that each particle lives in a heat bath which includes some of its own radiation.

It is certainly of interest to note that the 2.7 K background radiation is found to pervade the whole universe. The above would lead us to suggest that this radiation is caused by the average of the motion of all the particles in the universe. Furthermore, this would provide a simple resolution for two of Rowan-Robinson's 20 controversies in cosmology without inflation (Rowan-Robinson, 1996).

The first controversy (Rowan-Robinson, #4), is related to the high degree of isotropy of the microwave background radiation on all scales (the limit of anisotropy is 10^{-3}%). The measured speed of our galaxy through this radiation is about 600 km s^{-1}, and it provides an isotropic frame of reference at every point in the universe. This implies that both the velocity and acceleration of this frame are fixed and hence goes much farther than general relativity allows (e.g., an infinite number of freely falling frames). Note that from Theorem 3.8 we have $d\tau = (Mc^2/H)dt$. Since (Mc^2/H) is assumed constant, we get $\tau(H/Mc^2) = t$, or $t = (U^2 + c^2)^{1/2}\,\tau$. Since τ is fixed, we see that U is fixed for all observers. However, even if the conditions of Theorem 3.8 are not satisfied, our analysis of the global fields (5.16) and (5.17) still provides a simple explanation for the background radiation and why it is so isotropic.

The second controversy (Rowan-Robinson, #6) is the horizon problem and inflation. "When we look at the microwave background radiation in two opposite directions on the sky, we are looking back at regions that, . . . , have never been in communication. . . . How did these regions come to be so similar to each other, satisfying homogeneity and isotropy to one part in 100,000?" In our interpretation, neither inflation nor the big bang model is required to explain the uniformity of the background radiation.

5.3. Time Arrow

Since $d\tau = (mc^2/H)dt$, $K = [H^2/2mc^2 + mc^2/2]$, we see that, if $t \to -t$ (time reversal), then $K \to K$ is invariant, while $\tau \to -\tau$. On the other hand, if $H \to -H$, then $mc^2 \to -mc^2$, so that $K \to -K$. In this case, our Poisson bracket is invariant, as $\mathbf{p} \to -\mathbf{p}$ (e.g., $\{,\} \to \{,\}$). In either case, the equations of motion are invariant, so that there is no gain in exploiting the two possible signs when taking the square root for equation (1.3). Thus, our particle theory is invariant under time reversal.

As noted earlier, the field equations carry intrinsic information about the velocity and acceleration of the particles. Since this information is about the past behavior of the particles, reversing the proper-time of the particles no longer corresponds to an equivalent physical process run backward. Such an operation now corresponds to specifying the future behavior of the velocity and acceleration of the particle variables (explicitly). Since the global fields deposit radiation thoughout the domain of the system, time reversal would correspond to radiation leaving each part of the domain and collecting on the particles. This is certainly mathematically possible, but does not represent physics as we know it. *It follows that we must use retarded fields and potentials so that the field equations introduce an arrow for time*. We note that the discovery of time-reversal noninvariance in K^0 decay is now over 30 years old (Kabir, 1968) and no satisfactory explanation has been found.

It is interesting that this very issue was the cause of a debate between Einstein and Ritz in 1908 (Fritzius, 1990). Einstein contended that one could use both the retarded and advanced potentials and that the time arrow is caused by considerations of a statistical nature. Ritz maintained that only retarded potentials are justified because the irreversibility of radiation processes is basic.

In our formulation, it is thus natural to interpret antimatter as matter with its proper-time reversed. If our theory is correct, there is no antimatter in our universe except that which is created by extreme conditions (e.g., black holes, supernovas, etc.) and its existence can be considered virtual. A complete discussion requires the introduction of Santilli's (1993b) isodual numbers in which the unit 1 is replaced by -1 and $ab \to a*b = -ab$, so that $(-1)*(-1) = -1$. This allows for a completely symmetric theory of matter (and numbers) which avoids all of the objections to hole theory. We will discuss this completely as a part of our approach to the foundations of relativistic quantum theory.

The arrow of historical time has been discussed extensively by Fanchi (1993); (see also Fanchi, 1987) and by Horwitz *et al* (1989). Both Feynman (1948) and Stückelberg (1942) introduced the notion of representing antimatter as matter with its time reversed. Our final conclusion is the same as theirs. However, the two approaches are distinct. In our approach, we replace t by τ and acquire K as its canonical Hamiltonian, so that τ becomes both our coordinate time and evolution parameter. In their approach, they retain t and H and introduce τ as an evolution parameter. This allows them to let $d/d\tau$ (in the four-vector sense) maintain the role of a Lorentz-invariant (operator-valued) quantity.

6. CONCLUSION

We have constructed a direct implementation of the first two postulates of the special theory of relativity that fixes the proper-time of the particle variables for all observers. This approach provides a natural generalization of Maxwell's equations which depends on the motion of the sources. The resulting wave equations contain a damping term and explains radiation reaction as inertial resistance to acceleration. It is shown that (in our approach) the speed of light depends on the motion of the source as suggested by Ritz (1908) and is not in contradiction with the second postulate. The dependence of the field equations on the past motion of the sources makes the theory noninvariant under time reversal, producing an arrow for time at the classical level.

It should also be clear that the Minkowski decision to use proper-time as a parameter and and to implement the two postulates of Einstein using four-vectors is a convention which does not uniquely determine the theory. We now have two distinct approaches to the implementation of Einstein's two postulates. In the Minkowski case, an additional postulate is required. In the proper-time approach, an additional postulate of a completely different nature is required; namely, that the local geometry changes isotopically when interaction is turned on. This approach allows us to solve in a direct way a number of unsolvable and/or intractable problems in the Minkowski approach. Although there is much to be done, it is clear that we have a general framework that allows us to think about problems in the special theory of relativity that are closer to the way physical reality appears to our consciousness. In Table I, we compare the Minkowski and the proper-time implementations.

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Table I. A Comparison of the Proper-Time and the Minkowski Approach

	Minkowski	Proper-time
Reference system	No special one required	No special one required
Light velocity	Independent of source	Depends on source
Spacetime	Dependent variables	Independent variables
Transformations	Lorentz, Lorentz[a]	Lorentz proper time[a]
Cluster property	No general proof	Follows from theory
Many-particle	No general theory	General theory
Radiation reaction	Problematic	Follows from theory
Quantum theory	Problematic	Directly possible
Time arrow	Problematic	Follows from theory
Universal clock	No theory	Follows from theory

[a] This means that the first transformation is between observers about the underlying geometric manifold, while the second transformation is between observers about the dynamical laws of an observed system.

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